INDECOMPOSABILITY OF VARIOUS PROFINITE GROUPS ARISING FROM HYPERBOLIC CURVES

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Abstract. In this paper, we prove that the étale fundamental group of a hyperbolic curve over an arithmetic field [e.g., a finite extension field of $\mathbb{Q}$ or $\mathbb{Q}_p$] or an algebraically closed field satisfies the indecomposability [i.e., cannot be decomposed into the direct product of nontrivial profinite groups]. Moreover, in the case of characteristic zero, we also prove that the étale fundamental group of the configuration space of a curve of the above type is indecomposable. Finally, we consider the topic of indecomposability in the context of the theory of combinatorial anabelian geometry and pose the question: Is the Grothendieck-Teichmüller group GT indecomposable? We give an affirmative answer to a pro-l version of this question.

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Introduction

In this paper, we study the indecomposability of various profinite groups. The term indecomposability is defined as follows [cf. Definition 1.1]:

> We shall say that a profinite group G is *indecomposable* if, for any isomorphism of profinite groups $G \cong G_1 \times G_2$, where G_1, G_2 are profinite groups, it follows that either G_1 or G_2 is the trivial group.

In the "zero-dimensional" case, i.e., the case of the absolute Galois group G_k of a field k, the following fact is known [cf. Theorem 1.2]:

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Fact. (Haran-Jarden [cf. [6], Corollary 2.5]) *Let k be a Hilbertian field [cf. [FJ], Chapter 12]. Then G_k is indecomposable.*

In particular, the absolute Galois group of a finitely generated (respectively, finitely generated transcendental) extension field of $\mathbb{Q}$ (respectively, $\mathbb{Q}_p$ or $\mathbb{F}_p$) is indecomposable [cf. Corollary 1.4]. Note that any p-adic local field [i.e., a finite extension field of $\mathbb{Q}_p$] is non-Hilbertian [cf. Remark 1.3], but whose absolute Galois group is also indecomposable [cf. Proposition 1.6].

In this paper, we treat the "positive-dimensional" case. In the following, for a connected noetherian scheme $(-)$, we shall write $\Pi_{(-)}$ for the étale fundamental group of $(-)$ [for some choice of basepoint]. Now we consider the case of étale fundamental groups of smooth [hyperbolic] curves. First, we prove the following theorem [cf. Theorems 2.1, 2.2] which concerns the case where the base field is algebraically closed.

Theorem A. *Let k be an algebraically closed field; X a smooth curve of type (g, r) over k such that the pair (g, r) satisfies $2g - 2 + r > 0$ (respectively, $(g, r) \neq (0, 0)$, $(1, 0)$) if the characteristic of k is zero (respectively, positive). Then Π_X is indecomposable.*

The characteristic zero case of Theorem A is shown in [20], Proposition 3.2.

Next, we consider the case that the base field is non-algebraically closed. Let k be a field of characteristic $p \geq 0$; $l \neq p$ a prime number. Then for the pair (k, l), we consider the following condition:

$(*^l_k)$ For any finite extension field k' of k, the l-adic cyclotomic character $\chi_{k'} : G_{k'} \to \mathbb{Z}_l^\times$ of k' is nontrivial.

We shall say that k is *l-cyclotomically full* if the pair (k, l) satisfies the condition $(*^l_k)$ [cf. Definition 3.1].

Then we prove the following theorem [cf. Theorem 3.3]:

Theorem B. *Let k be a field of characteristic $p \geq 0$ such that G_k is center-free and indecomposable; X a smooth curve of type (g, r) over k such that the pair (g, r) satisfies $2g - 2 + r > 0$ (respectively, $(g, r) \neq (0, 0)$, $(1, 0)$) if the characteristic of k is zero (respectively, positive). Suppose that there exists a prime number $l \neq p$ such that k is l-cyclotomically full. Then Π_X is indecomposable.*

Next, in the case of the étale fundamental group of the configuration space of a hyperbolic curve, we prove the following [cf. Theorem 3.4]:

Theorem C. *Let n be a positive integer; k a field of characteristic zero such that G_k is center-free and indecomposable; X a hyperbolic curve over k; X_n the n-th configuration space associated to X. Suppose that either k is algebraically closed, or l-cyclotomically full for a prime number l. Then Π_{X_n} is indecomposable.*

For instance, Theorems B and C imply the following [cf. Corollary 3.7]:

Corollary D. *Let n be a positive integer; k a field; X a smooth curve of type (g, r) over k such that the pair (g, r) satisfies $2g - 2 + r > 0$ (respectively, $(g, r) \neq (0, 0), (1, 0)$) if the characteristic of k is zero (respectively, positive); X_n the n-th configuration space associated to X. Then the following hold:*

(i) *If k is a finitely generated transcendental extension field of $\mathbb{F}_p$, then Π_X is indecomposable.*

(ii) *If k is a finitely generated extension field of either $\mathbb{Q}$ or $\mathbb{Q}_p$, then Π_{X_n} is indecomposable.*

Theorem C also implies the following geometric result [cf. Theorem 3.8]:

Theorem E. *Let n be a positive integer; k a field of characteristic zero; X a hyperbolic curve over k; X_n the n-th configuration space associated to X. Suppose that there exists an isomorphism of k-schemes*

$$X_n \xrightarrow{\sim} Y \times_k Z$$

— where Y, Z are k-varieties [i.e., schemes that are of finite type, separated, and geometrically integral over k]. Then it follows that either

$$Y \cong \mathrm{Spec}(k) \quad \textit{or} \quad Z \cong \mathrm{Spec}(k).$$

Finally, we consider the Grothendieck-Teichmüller group GT [cf. Definition 4.1]. One fundamental problem in the theory of GT is the issue of whether or not the well-known injection

$$G_{\mathbb{Q}} \hookrightarrow \mathrm{GT}$$

is, in fact, bijective. On the other hand, from the point of view of the theory of combinatorial anabelian geometry [cf., e.g., [18], [10], [11], [12]], we recall that it is stated in [12], Introduction, that:

> "By contrast, one important theme of the present series of papers lies in the point of view that, instead of pursuing the issue of whether or not GT is literally isomorphic to $G_{\mathbb{Q}}$, it is perhaps more natural to concentrate on the issue of verifying that GT exhibits analogous behavior/properties to $G_{\mathbb{Q}}$ [or $\mathbb{Q}$]."

From this point of view, it is natural to pose the following question:

Is GT indecomposable?

[Note that $G_{\mathbb{Q}}$ is indecomposable [cf. the above **Fact**].] In this paper, we give an affirmative answer to a pro-l version of this question. More precisely, we prove the following result [cf. Theorem 4.4]:

Theorem F. *Let l be a prime number. Then the pro-l Grothendieck-Teichmüller group GT_l [cf. Definition 4.1] is indecomposable.*

The present paper is organized as follows: In §1, we review various properties of absolute Galois groups. Also, we prove a [profinite] group-theoretic result [cf. Proposition 1.7] which is needed in §3. In §2, we prove the indecomposability of the geometric fundamental group of a smooth [hyperbolic] curve [cf. Theorem A]. In §3, by applying the results of §1 and §2, we prove

Theorems B, C and Corollary D. Moreover, as an application of Theorem C, we conclude Theorem E. Finally, in §4, after reviewing the definitions of GT and GT_l, we verify Theorem F as a consequence of a certain anabelian result over finite fields [cf. [7], Remark 6, (iv)].

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0. Notations and Conventions

In this paper, we follow the terminology and conventions of [20], §0, "Topological Groups", "Curves"; [19], Definition 1.1, (ii), (iii).

Fields: A finite extension field of $\mathbb{Q}$ (respectively, $\mathbb{Q}_p$) will be referred to as a *number field* (respectively, *p-adic local field*).

Topological groups: Let G be a Hausdorff topological group, and $H \subseteq G$ a closed subgroup. Let us write $Z_G(H)$ for the *centralizer* of H in G. We shall write $Z(G) \stackrel{\mathrm{def}}{=} Z_G(G)$ for the *center* of G.

We shall say that a profinite group G is *elastic* if it holds that every topologically finitely generated closed normal subgroup $N \subseteq H$ of an open subgroup $H \subseteq G$ of G is either trivial or of finite index in G. If G is elastic, but not topologically finitely generated, then we shall say that G is *very elastic*.

We shall say that a profinite group G is *slim* if for every open subgroup $H \subseteq G$, the centralizer $Z_G(H)$ is trivial. A profinite group G is slim if and only if every open subgroup of G has trivial center [cf. [16], Remark 0.1.3]. It is easily verified that every finite closed normal subgroup $N \subseteq G$ of a slim profinite group G is trivial.

Let p be a prime number. Then we shall write $G^{(p)}$ for the *maximal pro-p quotient* of a profinite group G. If G admits an open subgroup which is pro-p, then we shall say that G is *almost pro-p*.

We shall write G^{ab} for the *abelianization* of a profinite group G, i.e., the quotient of G by the closure of the commutator subgroup of G.

If G is a topologically finitely generated profinite group, then one verifies easily that the topology of G admits a basis of characteristic open subgroups. Any such basis determines a profinite topology on the groups $\mathrm{Aut}(G)$, $\mathrm{Out}(G)$.

Let X be a connected noetherian scheme. Then we shall write Π_X for the *étale fundamental group* of X [for some choice of basepoint]. For any field k, we shall write $G_k \stackrel{\mathrm{def}}{=} \Pi_{\mathrm{Spec}(k)}$ for the *absolute Galois group* of k.

Curves: Let S be a scheme and X a scheme over S. If (g, r) is a pair of nonnegative integers, then we shall say that $X \to S$ is a *smooth curve of type* (g, r) over S if there exist an S-scheme $\overline{X}$ which is smooth, proper, of relative dimension 1 with geometrically connected fibers of genus g, and a closed subscheme $D \subseteq \overline{X}$ which is finite étale of degree r over S such that the complement of D in $\overline{X}$ is isomorphic to X over S.

We shall say that X is a *hyperbolic curve* over S if there exists a pair (g, r) of nonnegative integers with $2g - 2 + r > 0$ such that X is a smooth curve of type (g, r) over S. A *tripod* is a hyperbolic curve of type $(0, 3)$.

Let $X \to S$ be a smooth curve of type (g, r), and P_n the fiber product of n copies of X over S. Then we shall refer to as the *n-th configuration space associated to* X the S-scheme X_n which represents the open subfunctor

$$T \mapsto \{(f_1, \ldots, f_n) \in P_n(T) \mid f_i \neq f_j \quad \text{if} \quad i \neq j\ \}$$

of the functor represented by P_n [cf. [7], Definition 2.1, (i), (ii)].

1. Indecomposability of Absolute Galois Groups

In this section, we review various properties of absolute Galois groups. Also, we prove a [profinite] group-theoretic result [cf. Proposition 1.7] which is needed in §3.

Definition 1.1. (cf. [20], Definition 3.1) We shall say that a profinite group G is *indecomposable* if, for any isomorphism of profinite groups $G \cong G_1 \times G_2$, where G_1, G_2 are profinite groups, it follows that either G_1 or G_2 is the trivial group. We shall say that G is *strongly indecomposable* if every open subgroup of G is indecomposable.

Theorem 1.2. *Let k be a Hilbertian field [cf. [FJ], Chapter 12]. Then G_k is very elastic, slim, and strongly indecomposable.*

Proof. The very elasticity portion of Theorem 1.2 follows from [4], Lemma 16.11.5; [4], Proposition 16.11.6. Note that for any open subgroup H of G_k, there exists a finite separable extension k_H of k such that $G_{k_H} \xrightarrow{\sim} H$. Here, by [4], Corollary 12.2.3, k_H is also a Hilbertian field. Thus, to verify the slimness and the strong indecomposability portions of Theorem 1.2, it suffices to show that G_k is center-free and indecomposable. But this center-freeness (respectively, indecomposability) follows from [4], Proposition 16.11.6 (respectively, the theorem of Haran-Jarden [cf. [6], Corollary 2.5]). $\square$

Remark 1.3. Let k be either a finite field or a p-adic local field. Then k is always non-Hilbertian. Indeed, G_k is topologically finitely generated [cf. Proposition 1.6, below; [4], Lemma 16.11.5].

Corollary 1.4. *The following types of fields are Hilbertian:*

(i) *finitely generated extension fields of* $\mathbb{Q}$,

(ii) *finitely generated transcendental extension fields of* $\mathbb{Q}_p$.

(iii) *finitely generated transcendental extension fields of* $\mathbb{F}_p$.

In particular, their absolute Galois groups are very elastic, slim, and strongly indecomposable.

Proof. The first statement follows from [4], Theorem 13.4.2. The last statement follows from the first, together with Theorem 1.2. ☐

Lemma 1.5. *Let G be a profinite group. If G is elastic, slim, and topologically finitely generated, then G is strongly indecomposable.*

Proof. First, we note that any open subgroup of G is also elastic, slim, and topologically finitely generated. Thus, to verify the assertion, it suffices to show that G is indecomposable. Suppose that we have an isomorphism of profinite groups $G \cong G_1 \times G_2$ such that $G_1 \neq \{1\}$. Then since G_1 is a nontrivial topologically finitely generated closed normal subgroup of G, [by the elasticity of G] G_1 is of finite index in G. In particular, G_1 is an open subgroup of G. Thus, by the slimness of G, we have $G_2 \subseteq Z_G(G_1) = \{1\}$. ☐

Proposition 1.6. *Let k be a p-adic local field. Then G_k is elastic, slim, topologically finitely generated, and strongly indecomposable.*

Proof. The assertions follow from Lemma 1.5; [19], Theorem 1.7, (ii); [21], Theorem 7.4.1. ☐

Proposition 1.7. *Let*

$$1 \longrightarrow \Delta \longrightarrow \Pi \xrightarrow{p} G \longrightarrow 1$$

be an exact sequence of profinite groups. Then the following hold:

(i) *Suppose that Δ is indecomposable, and G is center-free and indecomposable. Then if the natural outer Galois representation*

$$G \to \mathrm{Out}(\Delta)$$

associated to the above exact sequence is nontrivial, then Π is also indecomposable.

(ii) *Suppose that Δ is nontrivial and center-free, and that G is nontrivial. Then if Π is indecomposable, then the natural outer Galois representation*

$$G \to \mathrm{Out}(\Delta)$$

associated to the above exact sequence is nontrivial.

Proof. (i) Suppose that $\Pi = \Pi_1 \times \Pi_2$, where Π_1, Π_2 are nontrivial closed normal subgroups of Π. Then since G is center-free, it follows from [20], Proposition 3.3 that there exist normal closed subgroups $H_i \subseteq \Pi_i$ [for $i = 1$, 2] such that $\Pi_1/H_1 \times \Pi_2/H_2 \xrightarrow{\sim} G$. In particular, since G is indecomposable, we obtain that either $\Pi_1/H_1 = \{1\}$ or $\Pi_2/H_2 = \{1\}$. Without loss of generality, we may assume that $\Pi_1/H_1 = \{1\}$, so $\Pi_1 = H_1$, $\Pi_2/H_2 \xrightarrow{\sim} G$. Thus, we have $\Pi_1 \times H_2 \xrightarrow{\sim} \Delta$.

Now I claim that $H_2 \neq \{1\}$. Indeed, suppose that $H_2 = \{1\}$, so $\Pi_1 \xrightarrow{\sim} \Delta$, $\Pi_2 \xrightarrow{\sim} G$. Then the extension determined by the exact sequence that appears in the statement of Proposition 1.7 is isomorphic to the trivial extension of G by Δ

$$1 \longrightarrow \Delta \longrightarrow \Delta \times G \longrightarrow G \longrightarrow 1.$$

Thus, the natural outer Galois representation $G \to \mathrm{Out}(\Delta)$ induced by the conjugation action of G on Δ is trivial. But this contradicts the assumption that the outer representation $G \to \mathrm{Out}(\Delta)$ is nontrivial. This completes the proof of the claim.

In light of the claim, by the indecomposability of Δ, we conclude that $\Pi_1 = \{1\}$, a contradiction. This completes the proof that Π is indecomposable.

(ii) Suppose that the representation $G \to \mathrm{Out}(\Delta)$ is trivial. Here, note that both Δ and $Z_\Pi(\Delta)$ are normal closed subgroups of Π. Moreover, by the triviality of the representation $G \to \mathrm{Out}(\Delta)$, it follows that Π is generated by Δ and $Z_\Pi(\Delta)$. Thus, since Δ is center-free, i.e., $\Delta \cap Z_\Pi(\Delta) = Z(\Delta) = \{1\}$, we obtain that $\Pi \cong \Delta \times Z_\Pi(\Delta)$. Here, we note that since $p(Z_\Pi(\Delta)) = G$ is nontrivial, we have $Z_\Pi(\Delta) \neq \{1\}$. Therefore, since Δ is nontrivial, we conclude that Π is not indecomposable, a contradiction. □

2. Indecomposability of Geometric Fundamental Groups of Curves

In this section, we prove the indecomposability of the geometric fundamental group of a smooth [hyperbolic] curve.

Theorem 2.1. *Let k be an algebraically closed field of characteristic zero; X a hyperbolic curve over k. Then Π_X is elastic, slim, and topologically finitely generated. In particular, Π_X is strongly indecomposable.*

Proof. The fact that Π_X is elastic (respectively, slim; topologically finitely generated) follows from [20], Theorem 1.5 (respectively, [20], Proposition 1.4; [23], EXPOSÉ XIII, Corollaire 2.12). In particular, the strong indecomposability of Π_X follows from Lemma 1.5 [cf. also [20], Proposition 3.2; [20], Remark 3.2.1]. □

Theorem 2.2. *Let k be an algebraically closed field of characteristic $p > 0$; X a smooth curve of type (g, r) over k such that the pair (g, r) satisfies $(g, r) \neq (0, 0)$, $(1, 0)$. Then $G \stackrel{\mathrm{def}}{=} \Pi_X$ is strongly indecomposable.*

Proof. First, we note that for any open subgroup H of G, there exists a connected finite étale covering $X_H \to X$ of X, where X_H is also a curve of type $\neq (0, 0)$, $(1, 0)$ over k such that $\Pi_{X_H} \xrightarrow{\sim} H$. Thus, to verify the assertion, it suffices to show that G is indecomposable. Suppose that we have an isomorphism of profinite groups $G \cong G_1 \times G_2$ such that $G_1 \neq \{1\}$, $G_2 \neq \{1\}$. In particular, by the slimness of G [cf. Proposition 2.4, below], it follows that G_1, G_2 are infinite [cf. §0].

Now I claim that

($*_1$) there exists an open subgroup U of G such that U is [isomorphic to] the fundamental group of a curve of genus ≥ 2.

Indeed, this fact is elementary and well-known, but we give a short proof here for completeness. First, we consider the case where the genus of X is 0, i.e., the unique smooth compactification of X is $\mathbb{P}^1_k$. Here, note that if we identify the function field of $\mathbb{P}^1_k$ with $k(t)$, where t is an indeterminate, then for any Artin-Schreier equation

$$x^p - x = t^m \quad (m \in \mathbb{Z}_{>0},\ p \nmid m),$$

one computes easily that the normalization of $\mathbb{P}^1_k$ in the extension field $k(t)[x]/(x^p - x - t^m)$ of $k(t)$ determines a finite ramified covering $\phi_m : C_m \to \mathbb{P}^1_k$ of $\mathbb{P}^1_k$ branched only at ∞, where C_m is a smooth, proper curve of genus $\frac{(m-1)(p-1)}{2}$ [cf., e.g., [26], Example 8.16]. Thus, for any curve X of type $(0, r)$, where $r > 0$, by taking m to be sufficiently large, we obtain a connected finite étale covering $X' \to X$ of X such that the genus of X' is ≥ 2. Next, we consider the case where the genus of X is 1, i.e., the unique smooth compactification of X is an elliptic curve E. Note that by applying the Riemann-Roch Theorem to E, we obtain a finite morphism $E_1 \stackrel{\mathrm{def}}{=} E \setminus \{p\} \to \mathbb{A}^1_k$ over k, where $p \in E \setminus X$ is a closed point of E. Next, let us observe that it follows from the genus 0 case, which has already been verified, that there exists a connected finite étale covering $C \to \mathbb{A}^1_k$ of $\mathbb{A}^1_k$ such that the genus of C is ≥ 2. Then any connected component of $E_1 \times_{\mathbb{A}^1_k} C$ determines a connected finite étale covering $C' \to E_1$ of E_1. Moreover, by applying the Hurwitz formula to the compactification of the finite morphism $C' \hookrightarrow E_1 \times_{\mathbb{A}^1_k} C \to C$, it follows that the genus of C' is also ≥ 2. Thus, for any curve X of type $(1, r)$, where $r > 0$, we obtain a connected finite étale covering $X' \to X$ of X such that the genus of X' is ≥ 2. This completes the proof of ($*_1$).

In light of ($*_1$) and the fact that G_1, G_2 are infinite, we may assume, without loss of generality, that G is the fundamental group of a curve of genus ≥ 2.

Next, I claim that

($*_2$) for every prime number $l \neq p$, there exist finite quotients $G_1 \twoheadrightarrow Q_1$, $G_2 \twoheadrightarrow Q_2$ such that l divides the order of Q_1, Q_2.

Indeed, suppose that l does not divide the order of any finite quotient of G_1. Now let $N_1 \subsetneq G_1$ be a proper normal open subgroup of G_1. Note that by assumption, we have $N_1^{\mathrm{ab}} \otimes \mathbb{Z}_l = \{1\}$. Write $N \stackrel{\mathrm{def}}{=} N_1 \times G_2$. Then since the conjugation action of $G/N \cong G_1/N_1 \times \{1\}$ on

$$N^{\mathrm{ab}} \otimes \mathbb{Z}_l \cong (N_1^{\mathrm{ab}} \otimes \mathbb{Z}_l) \times (G_2^{\mathrm{ab}} \otimes \mathbb{Z}_l) \cong \{1\} \times (G_2^{\mathrm{ab}} \otimes \mathbb{Z}_l)$$

is trivial, by Proposition 2.4, below, we conclude that $G/N = \{1\}$, a contradiction. This completes the proof of ($*_2$).

In light of the ($*_2$), by replacing G by the maximal pro-l quotient of a suitable open subgroup of G for some $l \neq p$, we may assume without loss of generality that G, G_1, G_2 are pro-l groups. Then since G is slim [cf. Proposition 2.4, below], it follows that G_1, G_2 are nonabelian pro-l groups, so $\dim_{\mathbb{F}_l} H^1(G_1, \mathbb{F}_l) \geq 2$, $\dim_{\mathbb{F}_l} H^1(G_2, \mathbb{F}_l) \geq 2$ [cf. [22], Theorem 7.8.1]. In particular, since we have an inclusion $H^1(G_1, \mathbb{F}_l) \otimes H^1(G_2, \mathbb{F}_l) \subseteq H^2(G, \mathbb{F}_l)$, we obtain that $\dim_{\mathbb{F}_l} H^2(G, \mathbb{F}_l) \geq 4$. This contradicts the fact that $\dim_{\mathbb{F}_l} H^2(G, \mathbb{F}_l)$ is either 0 or 1. [Indeed, $H^2(G, \mathbb{F}_l)$ is isomorphic to the second étale cohomology group $H^2_{\text{ét}}(X, \mathbb{F}_l)$ of X [cf. [17], Proposition 1.1]; the dimension over $\mathbb{F}_l$ of this last cohomology group is either 0 or 1 [cf. [5], Theorem 7.2.9 (ii); Proposition 7.2.10].] Therefore, G is indecomposable. $\square$

Remark 2.3. In the situation of Theorem 2.2, if X is an affine curve, then Π_X is never finitely generated. [In fact, the maximal pro-p quotient of Π_X is a free pro-p group of rank $|k|$ — cf. [24], Theorem 12.] In particular, we cannot apply Lemma 1.5 to Theorem 2.2.

The following result is well-known [cf., e.g., [25], Proposition 1.11; [20], Propostion 1.4], but we review it briefly for the sake of completeness.

Proposition 2.4. *Let k be an algebraically closed field of characteristic $p \geq 0$; $l \neq p$ a prime number; X a smooth curve of type (g, r) over k such that the pair (g, r) satisfies $2g - 2 + r > 0$ (respectively, $(g, r) \neq (0, 0)$, $(1, 0)$) if the characteristic of k is zero (respectively, positive). Then for any normal open subgroup N of $G \overset{\mathrm{def}}{=} \Pi_X$ such that the connected finite étale covering $X_N \to X$ corresponding to N has genus ≥ 2, the conjugation action of G/N on $N^{\mathrm{ab}} \otimes \mathbb{Z}_l$ is faithful. In particular, Π_X, as well as its maximal pro-l quotient $\Pi_X^{(l)}$, is slim.*

Proof. The faithfulness portion of Proposition 2.4 follows immediately from the argument given in [3], Lemma 1.14. The slimness portion of Proposition 2.4 follows formally from the faithfulness portion of Proposition 2.4. $\square$

3. Indecomposability of Various Fundamental Groups

In this section, by applying the results of §1 and §2, we prove the indecomposability of various fundamental groups. Moreover, by applying an indecomposability result, we prove the "scheme-theoretic indecomposability" of the configuration space of a hyperbolic curve over a field of characteristic zero [cf. Theorem 3.8].

Definition 3.1. Let k be a field of characteristic $p \geq 0$; $l \neq p$ a prime number. Then for the pair (k, l), we consider the following condition:

$(*^l_k)$ For any finite extension field k' of k, the l-adic cyclotomic character $\chi_{k'} : G_{k'} \to \mathbb{Z}_l^\times$ of k' is nontrivial.

We shall say that k is *l-cyclotomically full* if the pair (k, l) satisfies the condition $(*^l_k)$.

Lemma 3.2. *In the notation of Definition 3.1, the following hold:*

(i) *Let l, p be two distinct prime numbers; $k \in \{\mathbb{Q}, \mathbb{Q}_l, \mathbb{Q}_p, \mathbb{F}_p\}$. Suppose that K is a finitely generated extension field of k. Then K is l-cyclotomically full.*

(ii) *Let X be a smooth curve of type (g, r) over k such that the pair (g, r) satisfies $(g, r) \neq (0, 0)$, $(0, 1)$ (respectively, $(g, r) \neq (0, 0)$) if the characteristic of k is zero (respectively, positive); $\overline{k}$ an algebraic closure of k. Write $X_{\overline{k}} \stackrel{\mathrm{def}}{=} X \times_k \overline{k}$. Suppose, moreover, that k is l-cyclotomically full. Then the image of the natural outer Galois representation*

$$\rho_k : G_k \to \mathrm{Out}(\Pi_{X_{\overline{k}}})$$

associated to the "homotopy exact sequence"

$$1 \longrightarrow \Pi_{X_{\overline{k}}} \longrightarrow \Pi_X \longrightarrow G_k \longrightarrow 1$$

[cf. [23], EXPOSÉ IX, Théorème 6.1] is infinite, hence, in particular, nontrivial. If, moreover, $(g, r) \neq (0, 1)$, then the image of the naturally induced pro-l outer Galois representation

$$\rho_k^{(l)} : G_k \to \mathrm{Out}(\Pi^{(l)}_{X_{\overline{k}}})$$

is infinite, hence, in particular, nontrivial.

Proof. Assertion (i) follows from the various definitions involved.

We consider assertion (ii). First, suppose that $(g, r) = (0, 1)$ [so $p > 0$]. Then observe that one verifies immediately — by considering a suitable Artin-Schreier covering of X as in the proof of Theorem 2.2 over a suitable finite extension of k and applying [8], Lemma 23, (i), (iii) — that the infiniteness [hence, in particular, the nontriviality] of the image of ρ_k follows from the corresponding infiniteness in the case of $g \geq 1$. Here, we note that, although, in [8], Lemma 23, "Δ" [in the notation of [8], Lemma 23] is assumed to be topologically finitely generated, one verifies immediately that this assumption is in fact unnecessary. Thus, in the remainder of the proof of assertion (ii), we may assume without loss of generality that $(g, r) \neq (0, 1)$. Next, observe that to verify the infiniteness of ρ_k, it suffices to verify the infiniteness of $\rho_k^{(l)}$. Moreover, by replacing k by a suitable finite extension of k, it suffices to verify that $\rho_k^{(l)}$ is nontrivial. Suppose that $\rho_k^{(l)}$ is trivial. First, we assume that $g \geq 1$. Write $J(\overline{X})$ for the Jacobian variety of the smooth compactification $\overline{X}$ of X, $T_l(J(\overline{X}))$ for the l-adic Tate module of $J(\overline{X})$. Then it follows that the natural l-adic Galois representation

$$G_k \to \mathrm{Aut}(T_l(J(\overline{X})))$$

associated to $J(\overline{X})$ is trivial. Then since, as is well-known [cf. the natural isomorphisms $\bigwedge^{2g} H^1_{\text{ét}}(\overline{X}_{\overline{k}}, \mathbb{Z}_l) \xrightarrow{\sim} H^{2g}_{\text{ét}}(\overline{X}_{\overline{k}}, \mathbb{Z}_l) \xrightarrow{\sim} \mathbb{Z}_l(-g)$ of $\mathbb{Z}_l[G_k]$-modules discussed in [14], Remark 15.5; [13], Theorem 11.1, (a)], the determinant of this representation is a positive power of the l-adic cyclotomic character of k, we conclude that some positive power of the l-adic cyclotomic character of k is trivial. But this contradicts to the condition $(*^l_k)$. Next, we assume that $g = 0$ and $r \geq 2$. Then since $r \geq 2$, we may identify $X_{\overline{k}}$ with an open subscheme of $\mathbb{A}^1_{\overline{k}} \setminus \{0\}$. Thus, by considering the maximal pro-l abelian quotient of $\Pi_{\mathbb{A}^1_{\overline{k}} \setminus \{0\}}$, we conclude that the l-adic cyclotomic character of k is trivial, a contradiction. [Here, we recall that $H^1_{\text{ét}}(\mathbb{A}^1_{\overline{k}} \setminus \{0\}, \mathbb{Z}_l) \cong \mathbb{Z}_l(-1)$.] □

Theorem 3.3. *Let k be a field of characteristic $p \geq 0$ such that G_k is center-free and indecomposable; X a smooth curve of type (g, r) over k such that the pair (g, r) satisfies $2g - 2 + r > 0$ (respectively, $(g, r) \neq (0, 0)$, $(1, 0)$) if the characteristic of k is zero (respectively, positive). Suppose that there exists a prime number $l \neq p$ such that k is l-cyclotomically full. Then Π_X is center-free and indecomposable.*

Proof. Let $\overline{k}$ be an algebraic closure of k; $X_{\overline{k}} \overset{\text{def}}{=} X \times_k \overline{k}$. Then by [23], EXPOSÉ IX, Théorème 6.1, we have the following "homotopy exact sequence"

$$1 \longrightarrow \Pi_{X_{\overline{k}}} \longrightarrow \Pi_X \longrightarrow G_k \longrightarrow 1.$$

In particular, since G_k and $\Pi_{X_{\overline{k}}}$ are center-free [cf. Proposition 2.4], it follows that Π_X is also center-free. Here, we note that both G_k and $\Pi_{X_{\overline{k}}}$ are indecomposable [cf. Theorems 2.1, 2.2]. Thus, since the natural outer Galois representation

$$G_k \to \mathrm{Out}(\Pi_{X_{\overline{k}}})$$

associated to the above sequence is nontrivial [cf. Lemma 3.2, (ii)], it follows from Proposition 1.7, (i), that Π_X is also indecomposable. □

Theorem 3.4. *Let n be a positive integer; k a field of characteristic zero such that G_k is center-free and indecomposable; X a hyperbolic curve over k; X_n the n-th configuration space associated to X. Suppose that either k is algebraically closed, or l-cyclotomically full for a prime number l. Then Π_{X_n} is center-free and indecomposable.*

Proof. First, we note that for $n \geq 1$, any projection morphism $X_n \to X_{n-1}$ of length one determines a natural exact sequence of profinite groups [cf. [20], Proposition 2.2, (i)]

$$1 \longrightarrow \Pi_{(X_n)_{\overline{x}}} \longrightarrow \Pi_{X_n} \longrightarrow \Pi_{X_{n-1}} \longrightarrow 1$$

— where $\overline{x}$ is a geometric point of X_{n-1}; we write $X_0 \overset{\text{def}}{=} \mathrm{Spec}(k)$; $(X_n)_{\overline{x}}$ denotes the fiber of $X_n \to X_{n-1}$ over $\overline{x}$. In particular, by applying induction on n, we conclude from Proposition 2.4 and Theorem 3.3 that Π_{X_n} is center-free. Here, we note that $\Pi_{(X_n)_{\overline{x}}}$ and Π_{X_1} are indecomposable [cf. Theorems 2.1, 3.3]. Moreover, it is well-known that the natural outer Galois representation

$$\Pi_{X_{n-1}} \to \mathrm{Out}(\Pi_{(X_n)_{\overline{x}}})$$

associated to the above exact sequence is nontrivial. [In the case where k is an algebraically closed field, the above representation is, in fact, injective — cf. [2], Theorem 1.] Thus, by induction on n, it follows from Proposition 1.7, (i), that Π_{X_n} is indecomposable. □

Corollary 3.5. *Let n be a positive integer; k a Hilbertian field of characteristic $p \geq 0$; X a smooth curve of type (g, r) over k such that the pair (g, r) satisfies $2g - 2 + r > 0$ (respectively, $(g, r) \neq (0, 0)$, $(1, 0)$) if the characteristic of k is zero (respectively, positive); X_n the n-th configuration space associated to X. Suppose that there exists a prime number $l \neq p$ such that k is l-cyclotomically full. Also, if $p > 0$, then we assume further that $n = 1$. Then Π_{X_n} is center-free and indecomposable.*

Proof. These assertions follow immediately from Theorems 1.2, 3.3, 3.4. □

Remark 3.6. The center-freeness asserted in Theorems 3.3, 3.4 and Corollary 3.5 holds even if one does not assume that k is l-cyclotomically full.

Corollary 3.7. *Let n be a positive integer; k a field; X a smooth curve of type (g, r) over k such that the pair (g, r) satisfies $2g-2+r > 0$ (respectively, $(g, r) \neq (0, 0)$, $(1, 0)$) if the characteristic of k is zero (respectively, positive); X_n the n-th configuration space associated to X. Then the following hold:*

(i) *If k is a finitely generated transcendental extension field of $\mathbb{F}_p$, then Π_X is center-free and indecomposable.*
(ii) *If k is a finitely generated extension field of either $\mathbb{Q}$ or $\mathbb{Q}_p$, then Π_{X_n} is center-free and indecomposable.*

Proof. First, we note that every field k which appears in Corollary 3.7 is l-cyclotomically full for some prime number l [cf. Lemma 3.2, (i)]. Thus, in the case that k is Hilbertian [cf. Corollary 1.4] (respectively, non-Hilbertian, i.e., p-adic local), the assertions follow from Corollary 3.5 (respectively, Proposition 1.6 and Theorem 3.4). □

Theorem 3.8. *Let n be a positive integer; k a field of characteristic zero; X a hyperbolic curve over k; X_n the n-th configuration space associated to X. Suppose that there exists an isomorphism of k-schemes*

$$X_n \xrightarrow{\sim} Y \times_k Z$$

— where Y, Z are k-varieties [i.e., schemes that are of finite type, separated, and geometrically integral over k]. Then it follows that either

$$Y \cong \mathrm{Spec}(k) \quad \text{or} \quad Z \cong \mathrm{Spec}(k).$$

Proof. We may assume that k is algebraically closed. Then to verify the assertion, it suffices to show that either $\dim(Y) = 0$ or $\dim(Z) = 0$. First, we note that by the Künneth formula [cf. [23], EXPOSÉ XIII, Proposition 4.6], there exists an isomorphism of profinite groups

$$\Pi_{X_n} \xrightarrow{\sim} \Pi_Y \times \Pi_Z.$$

Then since Π_{X_n} is indecomposable by Theorem 3.4, we may without loss of generality that $\Pi_Y = \{1\}$. Now we fix a k-rational point $z \in Z(k)$ of

Z. Then we obtain a closed immersion $Y \xrightarrow{\sim} Y \times_k \{z\} \hookrightarrow Y \times_k Z \xrightarrow{\sim} X_n$. Write $Y' \to Y$ for the [surjective] morphism obtained by normalizing Y. Here, if we assume that $\dim(Y) \geq 1$, then the composite $Y' \to Y \hookrightarrow X_n$ is nonconstant. Thus, since X_n is of LFG-type [cf. [9], Definition 2.5] by [9], Proposition 2.7, the image of the outer homomorphism $\Pi_{Y'} \to \Pi_{X_n}$ is infinite — a contradiction. Therefore, we conclude that $\dim(Y) = 0$. $\square$

4. Indecomposability of the Pro-l Grothendieck-Teichmüller Group

In this section, we verify the indecomposability of the pro-l Grothendieck-Teichmüller group GT_l [cf. Theorem 4.4] as a consequence of a certain anabelian result over finite fields [cf. [7], Remark 6, (iv)].

Definition 4.1. (cf. [18], Definition 1.11, (i)) Let l be a prime number; k an algebraically closed field of characteristic zero; X the tripod $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$ over k; X_2 the second configurartion space associated to X. Suppose that $\Pi_1 \in \{\Pi_X, \Pi_X^{(l)}\}$. Write

$$\Pi_2 \stackrel{\mathrm{def}}{=} \begin{cases} \Pi_{X_2}, & \text{if } \Pi_1 = \Pi_X, \\ \Pi_{X_2}^{(l)}, & \text{if } \Pi_1 = \Pi_X^{(l)}. \end{cases}$$

Then for $n = 1$, 2, we shall write

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \subseteq \mathrm{Out}(\Pi_n)$$

for the subgroup of $\mathrm{Out}(\Pi_n)$ consisting of FC-admissible outomorphisms of Π_n [cf. [18], Definition 1.1, (ii)];

$$\mathrm{Out}^{\mathrm{FCS}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$$

for the subgroup of $\mathrm{Out}(\Pi_n)$ consisting of FC-admissible outomorphisms of Π_n that commute with the outer modular symmetries [cf. [18], Definition 1.1, (vi)];

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_1)^{\Delta+} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$$

for the image of $\mathrm{Out}^{\mathrm{FCS}}(\Pi_2)$ via the natural injection $\mathrm{Out}^{\mathrm{FC}}(\Pi_2) \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$ induced by the first projection $X_2 \to X$ [cf. [18], Definition 1.11, (i); [18], Corollary 1.12, (ii); [18], Corollary 4.2, (i)]. We shall refer to

$$\mathrm{GT} \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{FC}}(\Pi_X)^{\Delta+} \quad (\text{respectively, } \mathrm{GT}_l \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{FC}}(\Pi_X^{(l)})^{\Delta+})$$

as the *Grothendieck-Teichmüller group* (respectively, *pro-l Grothendieck-Teichmüller group*).

Remark 4.2. GT as defined in Definition 4.1 coincides with the Grothendieck-Teichmüller group as defined in more classical works [cf. [18], Remark 1.11.1].

The following lemma is well-known.

Lemma 4.3. *Let l be a prime number. Then* GT, GT_l *are slim.*

Proof. The asserted slimness follows immediately from the [pro-l] Grothendieck Conjecture over number fields [i.e., [15], Theorem A, applied to a tripod over a number field] and [11], Lemma 3.5. $\square$

Theorem 4.4. *Let l be a prime number. Then* GT_l *is strongly indecomposable.*

Proof. To verify the assertion, it suffices to show that for any open subgroup $U \subseteq \mathrm{GT}_l$ of GT_l, U is indecomposable. Let F be a finite field of characteristic $\neq l$. Write Δ for the maximal pro-l quotient of the étale fundamental group of the tripod $\mathbb{P}^1_{\overline{F}} \setminus \{0, 1, \infty\}$ over $\overline{F}$, where $\overline{F}$ is an algebraic closure of F, and

$$\rho : G_F \to \mathrm{Out}(\Delta)$$

for the pro-l outer Galois representation associated to $\mathbb{P}^1_F \setminus \{0, 1, \infty\}$. It follows immediately from the various definitions involved that $G \stackrel{\mathrm{def}}{=} \rho(G_F)$ is contained in $\mathrm{GT}_l \subseteq \mathrm{Out}(\Delta)$. Thus, by replacing F by a suitable finite extension of F, we may assume without loss of generality that $G \subseteq U$. Moreover, since $\mathrm{Out}(\Delta)$ is almost pro-l [cf. [1], Corollary 7], by replacing F by a suitable finite extension of F, we may assume without loss of generality that ρ factors through the maximal pro-l quotient $G_F \twoheadrightarrow G_F^{(l)}$ of G_F. Here, note that since G is infinite [cf. Lemma 3.2, (i), (ii)], we have $G \cong \mathbb{Z}_l$.

Now suppose that we have an isomorphism of profinite groups $U \cong H_1 \times H_2$. In the following, we shall identify U and $H_1 \times H_2$ via this isomorphism. Then I claim that it holds that

$$\text{either} \quad G \cap H_1 \neq \{1\} \quad \text{or} \quad G \cap H_2 \neq \{1\}.$$

Indeed, suppose that $G \cap H_1 = \{1\}$ and $G \cap H_2 = \{1\}$. In particular, it follows that, for $i = 1, 2$, the composite

$$G \hookrightarrow U = H_1 \times H_2 \stackrel{\mathrm{pr}_i}{\twoheadrightarrow} H_i$$

— where pr_i is i-th projection — is injective. Thus, if we write $K_i \subseteq H_i$ for the image of the above composite, we obtain that $G \stackrel{\sim}{\to} K_i$ [$\cong \mathbb{Z}_l$]. Here, note that we have inclusions

$$G \subseteq K \stackrel{\mathrm{def}}{=} K_1 \times K_2 \subseteq H_1 \times H_2.$$

Thus, since K [$\cong \mathbb{Z}_l \times \mathbb{Z}_l$] is abelian, we obtain that

$$K \subseteq Z_{\mathrm{GT}_l}(G) \hookrightarrow \mathbb{Z}_l^\times$$

— where "$\hookrightarrow$" is induced by the morphism "$\deg_{\mathbb{P}}$" of [7], Definition 3.1, which is injective by [7], Remark 6, (iv); [11], Lemma 3.5. In particular, by considering a suitable open subgroup of K, we obtain that $\mathbb{Z}_l \times \mathbb{Z}_l \cong \mathbb{Z}_l$, a contradiction. This completes the proof of the claim.

In light of the claim, we may assume without loss of generality that

$$G \cap H_1 \neq \{1\}.$$

Then since $G \cap H_1 \subseteq G$ is a nontrivial closed subgroup of $G \cong \mathbb{Z}_l$, it follows that $G \cap H_1$ is open in G. Thus, by replacing F by a suitable finite extension, we may assume without loss of generality that $G \subseteq H_1$. In particular, we obtain that

$$H_2 \subseteq Z_{\mathrm{GT}_l}(G) \hookrightarrow \mathbb{Z}_l^\times$$

— where "$\hookrightarrow$" denotes the arrow "$\hookrightarrow$" in the final display of the proof of the above claim. Thus, it follows that H_2 is abelian. On the other hand, since H_2 is center-free [cf. Lemma 4.3], we obtain that $H_2 = \{1\}$. Therefore, we conclude that U is indecomposable, as desired. □

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